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Gaussian effective potential and Coleman's normal-ordering prescription: the functional integral formalism

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Abstract

For a class of systems, the potential of whose bosonic Hamiltonian has a Fourier representation in the sense of tempered distributions, we calculate the Gaussian effective potential within the framework of the functional integral formalism. We show that Coleman's normal-ordering prescription can be formally generalized to the functional integral formalism.

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Recently, one of the authors, Lu, with his other collaborators obtained formulae of the Gaussian effective potential (GEP) [1] for a relatively general scalar field theory (see equation (1)) in the functional Schrödinger picture [2]. There, Coleman's normal-ordering prescription [3] was used, and accordingly these formulae have no divergences in low dimensions. Employing these formulae, one can obtain the GEP of any system in a certain class of models, which will be specified below, by carrying out ordinary integrations without performing functional integrations. In this paper, we demonstrate that the same formulae of the GEP can also be obtained within the functional integral formalism. In doing so, we also show that, although quantities in the functional integral formalism are not operators, Coleman's normal-ordering prescription can be formally used for renormalizing the GEP in the cases of low dimensions. We believe that our simple work is interesting and useful since the functional integral formalism is important in quantum field theory, nuclear and condensed matter physics [4], and can be used for performing some variational perturbation schemes [5, 6].

In this paper, we first generalize Coleman's normal-ordering prescription to the functional integral formalism. This formal generalization will be realized by borrowing the normal-ordered Hamiltonian expression in the functional Schrödinger picture because the Euclidean

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action for a system has the same form as the corresponding classical Hamiltonian in Minkowski space. Then following the procedure in [6], we calculate the GEP for a class of systems. Finishing the above generalization, as an explicit illustration, we will perform a model calculation for the $\lambda\phi^4$ field theory.

Consider a class of systems, scalar field systems or Fermi field systems which can be bosonized with the Lagrangian density

$$\mathcal{L}_x = \tfrac{1}{2}\partial_\mu\phi_x\partial^\mu\phi_x - V(\phi_x) \tag{1}$$

where the subscript x represents $x = (\vec{x}, t)$, the coordinates in a $(D+1)$-dimensional Minkowski space, ∂_μ and ∂^μ are the corresponding covariant derivatives, and ϕ_x the scalar field at x. In equation (1), the potential $V(\phi_x)$ has a Fourier representation in the sense of tempered distributions [7]. Speaking roughly, this requires that the integral $\int_{-\infty}^{\infty} V(\alpha)\,\mathrm{e}^{-C\alpha^2}\,\mathrm{d}\alpha$ with a positive constant C is finite. Obviously, quite a number of model potentials, such as polynomial models, sine-Gordon and sinh-Gordon models, possess this property.

For the system of equation (1), the conjugate field momentum is expressed as $\Pi_x \equiv \frac{\partial\mathcal{L}}{\partial(\partial_t\phi_x)} = \partial_t\phi_x$ and the Hamiltonian density is given by

$$\mathcal{H}_x = \tfrac{1}{2}\partial_t\phi_x\partial_t\phi_x + \tfrac{1}{2}\partial_{\vec{x}}\phi_x\partial_{\vec{x}}\phi_x + V(\phi_x). \tag{2}$$

In a time-fixed functional Schrödinger picture at $t = 0$, one can normal-order the Hamiltonian density $\mathcal{H}_x$ with respect to any given mass-dimension constant M as follows [2, 3][2]:

$$\mathcal{N}_M[\mathcal{H}_{\vec{x}}] = \tfrac{1}{2}\partial_t\phi_x\partial_t\phi_x + \tfrac{1}{2}\partial_{\vec{x}}\phi_{\vec{x}}\partial_{\vec{x}}\phi_{\vec{x}} + \mathcal{N}_M[V(\phi_{\vec{x}})] - \tfrac{1}{2}I_0[M^2] + \tfrac{1}{4}M^2I_1[M^2] \tag{3}$$

where $\mathcal{N}_M[\cdots]$ means the normal-ordering form with respect to M and

$$I_n[Q^2] = \int \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{\sqrt{p^2+Q^2}}{(p^2+Q^2)^n}. \tag{4}$$

Noticing the Baker–Haussdorf formula $\mathrm{e}^{A+B} = \mathrm{e}^A\,\mathrm{e}^B\,\mathrm{e}^{-\frac{1}{2}[A,B]}$, with the commutator $[A, B]$ some c-number, one has

$$\mathcal{N}_M[V(\phi_{\vec{x}})] = \int \frac{\mathrm{d}\Omega}{\sqrt{2\pi}}\tilde{V}(\Omega)\mathrm{e}^{\mathrm{i}\Omega\phi_{\vec{x}}}\,\mathrm{e}^{\frac{1}{4}\Omega^2I_1[M^2]} \tag{5}$$

where $\tilde{V}(\Omega)$ is the Fourier component of the function $V(\alpha)$. In the functional Schrödinger picture, when the Coleman's normal-ordering prescription [3] is adopted to calculate the GEP, i.e., when the normal-ordering Hamiltonian density takes the place of Hamiltonian density, the GEP will be naturally finite for the case of low dimensions ($D < 3$) [2].

On the other hand, for equation (1), the generating functional for Green's functions in the functional integral formalism is given by

$$Z_M[J] = \int \mathcal{D}\phi\,\exp\left\{\mathrm{i}\int \mathrm{d}^D\vec{x}\;\mathrm{d}t[\mathcal{L}_x + J_x\phi_x]\right\} \tag{6}$$

where J_x is an external source at x and $\mathcal{D}\phi$ the functional measure. Starting from equation (6), one can define the effective potential [8]. However, in this functional integral, the integrand is oscillatory. To avoid this oscillation, one usually adopts the so-called ϵ-prescription, i.e. in the Lagrangian density, one adds an infinitesimal purely imaginary term, $\mathrm{i}\epsilon\phi_x^2$, and takes $\epsilon \to 0$ after finishing the functional integration. Instead of doing so, one can also make the time continuation $t \to -\mathrm{i}\tau$ so that Minkowski space can be transformed into Euclidean space, and after finishing the functional integration, one may return to Minkowski space. This time continuation procedure is equivalent to the ϵ-prescription [8]. In this paper, we will choose the continuation procedure.

[2] Here, the partial derivative with time $\partial_t\phi_x = (\partial_t\phi_x)|_{t=0}$ should be regarded as the conjugate momentum operator $\Pi_{\vec{x}}$. For convenience of later comparison, we write the operator as its corresponding classical form.

Through the time continuation $t \to -\mathrm{i}\tau$, the generating functional $Z_M[J]$ is changed as

$$Z[J] = \int \mathcal{D}\phi \exp\left\{-\int \mathrm{d}^\nu r \left[\tfrac{1}{2}\partial_\tau \phi_r \partial_\tau \phi_r + \tfrac{1}{2}\partial_{\vec{x}} \phi_r \partial_{\vec{x}} \phi_r + V(\phi_r) - J_r \phi_r\right]\right\} \tag{7}$$

with $r = (\vec{x}, \tau)$ and $\nu = D + 1$. This is the generating functional in the Euclidean space. Here, we emphasize that in the above equation, taking the range of τ as $[0, \beta]$ with the inverse temperature β, letting J_r vanish and carrying out the functional integration over the closed path $\phi_r|_{\tau=0} = \phi_r|_{\tau=\beta}$, one can arrive at the canonical partition function of equation (1). From equation (7) one can get the generating functional for the connected Green's function, $W[J] = \ln(Z[J])$. The variational derivative of $W[J]$ with respect to J will give rise to the vacuum expectation value of the field ϕ_r in the presence of J_r,

$$\varphi_r = \frac{\delta W[J]}{\delta J_r}. \tag{8}$$

Taking a Legendre transformation of $W[J]$, one can define the effective potential in Euclidean space,

$$\mathcal{V}(\varphi) = -\left.\frac{W[J] - \int \mathrm{d}^\nu r J_r \varphi_r}{\int \mathrm{d}^\nu r}\right|_{\varphi_r = \varphi} \tag{9}$$

where φ is independent of the coordinate r. Returning to the Minkowski space from equation (9), one can get the effective potential in the Minkowski space, which is usually referred to as the effective potential in quantum field theory.

In the exponential of the functional integrand of equation (7), the major part of the integrand $\mathcal{H}_r = \frac{1}{2}\partial_\tau \phi_r \partial_\tau \phi_r + \frac{1}{2}\partial_{\vec{x}} \phi_r \partial_{\vec{x}} \phi_r + V(\phi_r)$ takes the same form as $\mathcal{H}_x$ in equation (2). Therefore, we argue that, if we change $\mathcal{H}_r$ in equation (7) into the expression[3] $\mathcal{N}_M[\mathcal{H}_r]$, the GEP will be renormalized automatically in the low dimensions ($D < 3$). Note that in the transformation between Euclidean space and Minkowski space, the integrals $I_{(n)}[Q^2]$ appearing in the functional integrations in Euclidean space,

$$I_{(n)}[Q^2] = \begin{cases} \displaystyle\int \frac{\mathrm{d}^\nu p}{(2\pi)^\nu} \frac{1}{(p^2 + Q^2)^n} & \text{for} \quad n \neq 0 \\ \displaystyle\int \frac{\mathrm{d}^\nu p}{(2\pi)^\nu} \ln(p^2 + Q^2) & \text{for} \quad n = 0 \end{cases} \tag{10}$$

are equivalent to $I_n[Q^2]$ in equation (4) which appear in the calculations in the functional Schrödinger picture (up to some constant factor or an infinite constant for some n) [6]. For example, $I(0)[Q^2]$ is equivalent to $I_0[Q^2]$ (up to an infinite constant) and $2I_{(1)}[Q^2]$ to $I_1[Q^2]$ [6]. Thus, corresponding to equation (3), one can formally write down $\mathcal{N}_M[\mathcal{H}_r] = \frac{1}{2}\partial_\tau \phi_r \partial_\tau \phi_r + \frac{1}{2}\partial_{\vec{x}} \phi_r \partial_{\vec{x}} \phi_r + \mathcal{N}_M[V(\phi_r)] - \frac{1}{2} I_{(0)}[M^2] + \frac{1}{2} M^2 I_{(1)}[M^2]$. Changing $\mathcal{H}_r$ in equation (7) into the form of $\mathcal{N}_M[\mathcal{H}_r]$, we have

$$\begin{aligned} Z[J] = \exp&\left\{\int \mathrm{d}^\nu r \left[\frac{1}{2} I_{(0)}[M^2] - \frac{1}{2} M^2 I_{(1)}[M^2]\right]\right\} \\ &\times \int \mathcal{D}\phi \exp\left\{-\int \mathrm{d}^\nu r \left[\frac{1}{2}\partial_\tau \phi_r \partial_\tau \phi_r + \frac{1}{2}\partial_{\vec{x}} \phi_r \partial_{\vec{x}} \phi_r \right.\right. \\ &\left.\left. - J_r \phi_r + \int \frac{\mathrm{d}\Omega}{\sqrt{2\pi}} \tilde{V}(\Omega) \mathrm{e}^{\mathrm{i}\Omega\phi_r} \, \mathrm{e}^{\frac{1}{2}\Omega^2 I_{(1)}[M^2]}\right]\right\} \\ = \exp&\left\{\int \mathrm{d}^\nu r \left[\frac{1}{2} I_{(0)}[M^2] - \frac{1}{2} M^2 I_{(1)}[M^2]\right]\right\} \int \mathcal{D}\phi \exp\{-S[J]\} \end{aligned} \tag{11}$$

[3] Note that $\mathcal{H}_x$ in equation (3) is an operator in D-dimensional space at $t = 0$, whereas the field ϕ_r and its derivatives in equation (7) are classical ones in ν-dimensional Euclidean space. This is why we call the generalization developed in the present paper a formal generalization.

where $S[J]=\int \mathrm{d}^{\nu}r\big[\frac{1}{2}\phi_r(-\nabla_r^2)\phi_r - J_r\phi_r + \int \frac{\mathrm{d}\Omega}{\sqrt{2\pi}}\tilde{V}(\Omega)\,\mathrm{e}^{\mathrm{i}\Omega\phi_r}\,\mathrm{e}^{\frac{1}{2}\Omega^2 I_{(1)}[M^2]}\big]$ with ∇_r the gradient with respect to r in ν-dimensional Euclidean space. Up to here, we have introduced Coleman's normal-ordering prescription in the functional integral formalism. Actually, many years ago, the normal-ordered Hamiltonian of the sine-Gordon field theory was used in the Euclidean functional integral formalism to show the equivalence between the sine-Gordon and massive Thirring field theories [9]. One will see that equation (11) will give rise to the same result in [2].

Now we calculate the GEP of equation (1) from equation (11) by using the procedure in [6]. For this purpose, $Z[J]$ will be modified through the following steps. First, a parameter μ is introduced by adding a vanishing term $\int \mathrm{d}^{\nu}r\frac{1}{2}\phi_r(\mu^2-\mu^2)\phi_r$ into $S[J]$. Then, shift ϕ_r to $\phi_r+\Phi$ with Φ a constant background field, i.e., $S[J]\to\int \mathrm{d}^{\nu}r\big[\frac{1}{2}\phi_r\big(-\nabla_r^2+\mu^2\big)\phi_r - J_r\phi_r - J_r\Phi + S_D\big]$ with $S_D=\int \mathrm{d}^{\nu}r\big[-\frac{1}{2}\mu^2\phi_r^2+\int\frac{\mathrm{d}\Omega}{\sqrt{2\pi}}\tilde{V}(\Omega)\,\mathrm{e}^{\mathrm{i}\Omega(\phi_r+\Phi)}\,\mathrm{e}^{\frac{1}{2}\Omega^2 I_{(1)}[M^2]}\big]$. Third, in the last resultant expression of $S[J]$, insert an expansion factor δ in front of S_D. Thus, $Z[J]$ is modified as the following $Z[J,\delta]$:

$$
\begin{aligned}
Z[J,\delta] &= \exp\left\{\int \mathrm{d}^{\nu}r\left[\frac{1}{2}I_{(0)}[M^2]-\frac{1}{2}M^2 I_{(1)}[M^2]+J_r\Phi\right]\right\}\\
&\quad\times\int\mathcal{D}\phi\exp\left\{-\int\mathrm{d}^{\nu}r\left[\frac{1}{2}\phi_r\left(-\nabla_r^2+\mu^2\right)\phi_r - J_r\phi_r\right]\right\}\exp\{-\delta S_D\} \qquad (12)\\
&= \left[\det\left(-\nabla_r^2+\mu^2\right)\right]^{-\frac{1}{2}}\exp\left\{\int\mathrm{d}^{\nu}r\left[\frac{1}{2}I_{(0)}[M^2]-\frac{1}{2}M^2 I_{(1)}[M^2]\right.\right.\\
&\quad\left.\left.+\frac{1}{2}\int\mathrm{d}^{\nu}r_1 J_r f^{-1}_{rr_1}J_{r_1}+J_r\Phi\right]\right\}\\
&\quad\times\frac{\int\mathcal{D}\phi\exp\big\{-\int\mathrm{d}^{\nu}r\left[\frac{1}{2}\phi_r\left(-\nabla_r^2+\mu^2\right)\phi_r - J_r\phi_r\right]\big\}\exp\{-\delta S_D\}}{\int\mathcal{D}\phi\exp\big\{-\int\mathrm{d}^{\nu}r\left[\frac{1}{2}\phi_r\left(-\nabla_r^2+\mu^2\right)\phi_r - J_r\phi_r\right]\big\}} \qquad (13)
\end{aligned}
$$

where $\det\big(-\nabla_r^2+\mu^2\big)$ is the determinant of $\big(-\nabla_r^2+\mu^2\big)$ and $f^{-1}_{rr_1}=\int\frac{\mathrm{d}^{\nu}p}{(2\pi)^{\nu}}\frac{1}{p^2+\mu^2}\,\mathrm{e}^{\mathrm{i}p\cdot(r-r_1)}$. In equation (13), the result of the Gaussian functional integral $\int\mathcal{D}\phi\exp\big\{-\int\mathrm{d}^{\nu}r\big[\frac{1}{2}\phi_r\big(-\nabla_r^2+\mu^2\big)\phi_r - J_r\phi_r\big]\big\}=\big[\det\big(-\nabla_r^2+\mu^2\big)\big]^{-\frac{1}{2}}\exp\big\{\frac{1}{2}\int\mathrm{d}^{\nu}r\mathrm{d}^{\nu}r_1 J_r f^{-1}_{rr_1}J_{r_1}\big\}$ has been used. Correspondingly, $W[J]$ is modified as $W[J,\delta]$. It is evident that, extrapolating $W[J,\delta]$ to $\delta=1$, one recovers $W[J]$. After the above modifications, expanding the logarithm of the functional integral in $W[J,\delta]=\ln Z[J,\delta]$ as a series in δ (i.e. expanding first $\mathrm{e}^{-\delta S_D}$ and then the logarithmic function), then truncating the series at the first order in δ, and finally carrying out the functional integrations, one has

$$
\begin{aligned}
W[J,\delta] &= \int\mathrm{d}^{\nu}r\left\{-\frac{1}{2}\left(I_{(0)}[\mu^2]-I_{(0)}[M^2]\right)-\frac{1}{2}M^2 I_{(1)}[M^2]+J_r\Phi+\frac{1}{2}\int\mathrm{d}^{\nu}r_1 J_r f^{-1}_{r_1 r}J_{r_1}\right.\\
&\quad+\delta\left[\frac{1}{2}\mu^2\left[I_{(1)}[\mu^2]+\left(\int\mathrm{d}^{\nu}r_1 f^{-1}_{rr_1}J_{r_1}\right)^2\right]\right.\\
&\quad\left.\left.-\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha}{\sqrt{\pi}}V\left(\alpha\sqrt{2\left(I_{(1)}[\mu^2]-I_{(1)}[M^2]\right)}+\int\mathrm{d}^{\nu}r_1 f^{-1}_{r_1 r}J_{r_1}+\Phi\right)\mathrm{e}^{-\alpha^2}\right]\right\} \qquad (14)
\end{aligned}
$$

where the first-order term of δ arises from the functional integral $\frac{\int\mathcal{D}\phi S_D\exp\{-\int\mathrm{d}^{\nu}r[\frac{1}{2}\phi_r(-\nabla_r^2+\mu^2)\phi_r - J_r\phi_r]\}}{\int\mathcal{D}\phi\exp\{-\int\mathrm{d}^{\nu}r[\frac{1}{2}\phi_r(-\nabla_r^2+\mu^2)\phi_r - J_r\phi_r]\}}$. In equation (14), we have used the integral formula $\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha}{\sqrt{2\pi}}\mathrm{e}^{-\frac{\alpha^2}{2}+\sqrt{2a}\alpha}=\mathrm{e}^{a}$ and the result $\big[\det\big(-\nabla_r^2+\mu^2\big)\big]^{-\frac{1}{2}}=\exp\big\{-\frac{1}{2}\int\mathrm{d}^{\nu}r I_0[\mu^2]\big\}$.

Therefore, up to the first order of δ, equation (8) gives

$$\varphi_r = \Phi + \int \mathrm{d}^\nu r_1 f^{-1}_{rr_1} J_{r_1} + \delta\mu^2 \int \mathrm{d}^\nu r_1 \mathrm{d}^\nu r_2 f^{-1}_{rr_1} f^{-1}_{r_1 r_2} J_{r_2} - \delta \int \mathrm{d}^\nu r_1 f^{-1}_{rr_1} \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha}{\sqrt{\pi}} V^{(1)} \times \left(\alpha \sqrt{2\left(I_{(1)}[\mu^2] - I_{(1)}[M^2]\right)} + \int \mathrm{d}^\nu r_1 f^{-1}_{rr_1} J_{r_1} + \Phi \right) \mathrm{e}^{-\alpha^2} \tag{15}$$

where $V^{(n)}(\alpha) = \frac{\mathrm{d}^n V(\alpha)}{(\mathrm{d}\alpha)^n} = \int \frac{\mathrm{d}\Omega}{\sqrt{2\pi}} \tilde{V}(\Omega)(\mathrm{i}\Omega)^n \mathrm{e}^{\mathrm{i}\Omega\alpha}$. In the last equation, one can take[4] $\varphi_r = \varphi = \Phi$ and hence solve it to get J_r in terms of Φ. This forces J_r to become a series in δ and vanish in the zeroth order of δ [6]. From equations (14) and (9), one can see that J_r truncated at the first order of δ has no contribution to $\mathcal{V}(\varphi)$ up to the first order of δ. Therefore, we have to take $J_r = 0$ for truncating equation (9) at the first order of δ and obtain the following result:

$$\mathcal{V}(\Phi) = \frac{1}{2}\left(I_{(0)}[\mu^2] - I_{(0)}[M^2]\right) + \frac{1}{2}M^2 I_{(1)}[M^2] - \frac{1}{2}\mu^2 I_{(1)}[\mu^2] + \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha}{\sqrt{\pi}} \mathrm{e}^{-\alpha^2} V\left(\alpha \sqrt{2\left(I_{(1)}[\mu^2] - I_{(1)}[M^2]\right)} + \Phi \right). \tag{16}$$

Obviously, the above equation is dependent on the arbitrary parameter μ. In accordance with the 'principle of minimal sensitivity' [6, 10], μ can be determined by requiring that μ should minimize $\mathcal{V}(\varphi)$. The stationary condition, $\frac{\partial \mathcal{V}(\varphi)}{\partial \mu^2} = 0$, yields

$$\mu^2(\varphi) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha}{\sqrt{\pi}} \mathrm{e}^{-\alpha^2} V^{(2)} \left(\alpha \sqrt{2\left(I_{(1)}[\mu^2] - I_{(1)}[M^2]\right)} + \Phi \right) \tag{17}$$

and the stability condition, $\frac{\partial^2 \mathcal{V}(\varphi)}{(\partial \mu^2)^2} \geqslant 0$, gives rise to

$$1 + \frac{1}{4} I_{(2)}[\mu^2] \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha}{\sqrt{\pi}} \mathrm{e}^{-\alpha^2} V^{(4)} \left(\alpha \sqrt{2\left(I_{(1)}[\mu^2] - I_{(1)}[M^2]\right)} + \Phi \right) \geqslant 0. \tag{18}$$

In order to investigate the symmetry-breaking phenomena, one usually needs another stationary point condition $\frac{\mathrm{d}\mathcal{V}(\varphi)}{\mathrm{d}\varphi} = 0$. This condition yields

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha}{\sqrt{\pi}} \mathrm{e}^{-\alpha^2} V^{(1)} \left(\alpha \sqrt{2\left(I_{(1)}[\mu^2] - I_{(1)}[M^2]\right)} + \Phi \right) = 0. \tag{19}$$

Noticing the equivalence between $I_{(n)}[Q^2]$ and $I_n[Q^2]$, and going back to the Minkowski space, equation (16) with equations (17) and (18) will give the GEP of the system, equation (1). We note that it is identical to that in [2]. We observe that, here, no renormalization procedure is needed for the case of $D < 3$, because the first three terms in equation (16) and $\left(I_{(1)}[\mu^2] - I_{(1)}[M^2]\right)$ are finite for $\mathrm{D} < 3$. As for the case of $D = 3$, the first three terms in equation (16) and $\left(I_{(1)}[\mu^2] - I_{(1)}[M^2]\right)$ are divergent, and so equations (16)–(19) have divergences. Hence, when $D = 3$, Coleman's normal-ordering prescription is not sufficient to renormalize the GEP, and further renormalization procedures are needed. In fact, Coleman's normal-ordering prescription amounts just to renormalizing the mass parameter. For the case of $D = 3$, one can further renormalize other model parameters and even the field to make the GEP finite.

By way of explanation and justification, we consider the $\lambda\phi^4$ field theory with the following potential:

$$V(\phi_x) = \frac{1}{2}m^2\phi_x^2 + \frac{\lambda}{4}\phi_x^4. \tag{20}$$

[4] Generally, different choices of φ will give rise to an identical result. One can find a detailed discussion on this point in appendix A of Stancu and Stevenson [6].

Employing the formulae $\int_{-\infty}^{\infty}\alpha^{2n}\mathrm{e}^{-\alpha^2}\mathrm{d}\alpha = 2^{-n}\cdot 1\cdot 3\cdot 5\cdots(2n-1)$ and $\int_{-\infty}^{\infty}\alpha^{2n+1}\mathrm{e}^{-\alpha^2}\,\mathrm{d}\alpha = 0$, one can easily finish the ordinary integrations over α in equations (16)–(19), and obtain

$$\begin{aligned}\mathcal{V}(\varphi) = {} & \frac{1}{2}\left(I_{(0)}[\mu^2] - I_{(0)}[M^2]\right) + \frac{1}{2}M^2 I_{(1)}[M^2] - \frac{1}{2}\mu^2 I_{(1)}[\mu^2] \\ & + \frac{1}{2}m^2\left(I_{(1)}[\mu^2] - I_{(1)}[M^2] + \Phi^2\right) + \frac{\lambda}{4}\Bigg[\frac{3}{4}\left(2I_{(1)}[\mu^2] - 2I_{(1)}[M^2]\right)^2 \\ & + 3\left(2I_{(1)}[\mu^2] - 2I_{(1)}[M^2]\right)\Phi^2 + \Phi^4\Bigg] \end{aligned} \tag{21}$$

$$\mu^2 = m^2 + 3\lambda\left(I_{(1)}[\mu^2] - I_{(1)}[M^2] + \Phi^2\right) \tag{22}$$

and

$$\frac{\mathrm{d}\mathcal{V}(\varphi)}{\mathrm{d}\varphi} = \Phi\left(m^2 + \frac{3\lambda}{2}\left(2I_{(1)}[\mu^2] - 2I_{(1)}[M^2]\right) + \lambda\Phi^2\right) = 0. \tag{23}$$

Recalling $I_{(0)}[Q^2] = I_0[Q^2]$ (up to an infinite constant) and $2I_{(1)}[Q^2] = I_1[Q^2]$, and noticing that for the case of (1 + 1) dimensions, $\frac{1}{2}\left(I_0[\mu^2] - I_0[M^2]\right) + \frac{1}{4}M^2 I_1[M^2] - \frac{1}{4}\mu^2 I_1[\mu^2] = \frac{\mu^2 - M^2}{8\pi}$ as well as $(I_1[\mu^2] - I_1[M^2]) = -\frac{1}{2\pi}\ln\frac{\mu^2}{M^2}$, one can find that equations (21) and (22) with $D = 1$ are consistent, respectively, with equations (A6) and (A7) for $B = 0$ in [3] (Chang) (the normal-ordering mass M was taken as m there and m' corresponds to μ here). Furthermore, the renormalized mass and coupling can be calculated as[5]

$$m_R^2 \equiv \left.\frac{\mathrm{d}^2\mathcal{V}(\Phi)}{\mathrm{d}\Phi^2}\right|_{\Phi=0} = m^2 + 3\lambda\left(I_{(1)}\left[m_R^2\right] - I_{(1)}[M^2]\right) \tag{24}$$

and

$$\lambda_R \equiv \left.\frac{1}{3!}\frac{\mathrm{d}^4\mathcal{V}(\Phi)}{\mathrm{d}\Phi^4}\right|_{\Phi=0} = \lambda\frac{1 - 6\lambda I_{(2)}\left[m_R^2\right]}{1 + 3\lambda I_{(2)}\left[m_R^2\right]} \tag{25}$$

respectively. The above expression of λ_R is consistent with equation (3.44) in [1] (1980) and equation (3.19) in [1] (1985), and has no explicit dependence upon the normal-ordering mass M (just an implicit dependence upon M through m_R). This fact implies that Coleman's normal-ordering prescription is involved only in the renormalization of the mass parameter. Because the integral $I_{(2)}$ is finite for the case of $D < 3$, the coupling does not require a further renormalization procedure. Substituting equation (24) into equations (21) and (22), one can get the GEP in terms of m_R instead of m and the resultant expressions for low dimensions are consistent with those in [1] (1985). Equation (24) reflects the relation between m and M_R, and has been discussed in detail for low dimensions in [11]. By the way, besides simplifying the renormalization procedure in low dimensions, Coleman's normal-ordering prescription makes it possible to investigate the symmetry restoration phenomenon in quantum field theory [11, 12]. As for the case of $D = 3$, both equations (24) and (25) are no longer finite relations, and further renormalization procedure will be needed to make the GEP finite. Stevenson and his collaborators have investigated this problem and proposed two non-trivial $\lambda\phi^4$ theories [13, 1 (1985)]. Based on Coleman's normal-ordering prescription, one of the present authors (Lu) gave a further discussion on Stevenson's two non-trivial $\lambda\phi^4$'s [14] (in [14] one can find many other references related to this problem).

In conclusion, we have demonstrated that Coleman's normal-ordering prescription can be formally used in the functional integral formalism to renormalize the GEP for a class of

[5] Here, the definition of the renormalized coupling is slightly different from that in [1], because the coupling there is four times that here.

systems in low dimensions. This conclusion will also be valid for the finite temperature GEP [15, 16]. Before ending this paper, we point out that the above renormalizability is understandable from the viewpoint of Feynman diagrams. Coleman's normal-ordering prescription can make ultraviolet divergences disappear in the theory whose primitively divergent graph is just the one-loop diagram with only one vertex. The (1 + 1)-dimensional scalar field theories without derivative interactions are just such theories [3]. Hence, the finiteness of equations (16)–(19) with $D = 1$ is conceivable. As for the case of $D = 2$, the additional primitively divergent graphs are two- or multi-loop diagrams with multi-vertices. These additional divergent diagrams are not included in the GEP, because the GEP is just the sum of all possible cactus diagrams [17, 1 (1980)] (a cactus diagram consists of one-loop diagrams with multi-vertices and/or loop diagrams with one vertex). And so the GEP in (2 + 1) dimensions can be made finite by Coleman's normal-ordering prescription. However, unfortunately, when $D = 3$, the one-loop diagram with two vertices, which comprises GEP, is divergent (for $D = 2$ such a diagram is finite), and so Coleman's normal-ordering prescription is not sufficient to make the (3 + 1)-dimensional GEP finite.

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